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Reflexive subspaces and lattices of pairs of projections

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Reflexive Subspaces and Lattices of Pairs of Projections

BY

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DISSERTATION

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This dissertation has been examined and approved.

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Dedication

To my husband.

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ABSTRACT

Reflexive Subspaces and Lattices of Pairs of Projections

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Consider the sets $\mathcal{P}_{\mathcal{H}}$ and $\mathcal{P}_{\mathcal{K}}$ of the projections onto closed subspaces of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. From the usual partial orders (based upon set containment) on $\mathcal{P}_{\mathcal{K}}$ and $\mathcal{P}_{\mathcal{H}}$, we can define a partial order on $\mathcal{P}_{\mathcal{K}} \times \mathcal{P}_{\mathcal{H}}$ by $(Q_1, P_1) \leq (Q_2, P_2)$ if and only if $P_1 \leq P_2$ and $Q_2 \leq Q_1$. Then the map $\alpha : \mathcal{P}_{\mathcal{K}} \times \mathcal{P}_{\mathcal{H}} \longrightarrow \mathcal{P}_{\mathcal{K}\oplus\mathcal{H}}$ given by $\alpha(Q, P) = (1-Q)\oplus P$ is an order-preserving map. In particular, if $\mathcal{L} \subseteq \mathcal{P}_{\mathcal{K}\times\mathcal{H}}$ is a lattice, then the restriction of α to $\mathcal{L}$ is a lattice isomorphism.

Let $\mathrm{Alg}(\mathcal{L})$ be the set of all operators taking each element of $\mathcal{L}$ into itself. If an operator $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in $B(\mathcal{H}, \mathcal{K})$ belongs to $\mathrm{Alg}(\alpha(\mathcal{L}))$, then for every pair (Q, P) in the lattice $\mathcal{L}$,

1. $QA(1 - Q) = 0$;

2. $QBP = 0$;

3. $(1 - P)C(1 - Q) = 0$; and

4. $(1 - P)DP = 0$.

The upper left coordinate of T must belong to $Alg(\{Q : (Q, P) \in \mathcal{L}\})$ and the lower right coordinate must belong to $Alg(\{P : (Q, P) \in \mathcal{L}\})$. The upper right coordinate belongs

to the subspace denoted below:

$$\mathcal{S}(\mathcal{L}) = \{T \in B(\mathcal{H}, \mathcal{K}) : QTP = 0 \ \forall \ (Q, P) \in \mathcal{L}\}.$$

In this paper, we will investigate the subspace $\mathcal{S}(\mathcal{L})$ for various types of lattices $\mathcal{L}$. Using the above map α, we will consider $\mathrm{Alg}(\alpha(\mathcal{L}))$ as a subspace of two by two operator-valued matrices with $\mathcal{S}(\mathcal{L})$ imbedded in the upper right corner.

We will consider what properties transfer between a subspace of operator-valued two by two matrices and the subspaces derived from each component; we may use these results to compare $\mathrm{Alg}(\alpha(\mathcal{L}))$ and $\mathcal{S}(\mathcal{L})$.

In the following sections, we examine particular types of lattices $\mathcal{L}$, namely nests and commutative subspace lattices. Many properties of the associated algebras can be used to describe the subspace $\mathcal{S}(\mathcal{L})$.

Chapter 1

Introduction

1.1 Preliminaries

In this chapter we first define notation and prove general results for later reference. Throughout this dissertation, $\mathcal{H}$ and $\mathcal{K}$ denote Hilbert spaces over the field of complex numbers (represented here by $\mathbf{C}$). The symbol $B(\mathcal{H})$ denotes the Banach algebra of bounded linear transformations from $\mathcal{H}$ to itself; $B(\mathcal{H}, \mathcal{K})$ denotes the space of bounded linear transformations from $\mathcal{H}$ to $\mathcal{K}$.

Unless otherwise stated, all subspaces will be assumed to be *closed* subspaces.

Structure theorems for operators can often be explained in terms of invariant subspaces.

Definition 1.1 *We say a subspace M of $\mathcal{H}$ is an invariant subspace for an operator T in $B(H, K)$ whenever $T(M) \subseteq M$. More generally, we say a subspace M of $\mathcal{H}$ is an invariant subspace for a set of operators $\mathcal{S}$ in $B(H, K)$ whenever $T(M) \subseteq M$ for every operator T in $\mathcal{S}$.*

If we let P represent the projection onto the subspace M, another way of stating that a subspace M is invariant under T is to say that $(1 - P)TP = 0$.

We know every complex-valued n by n matrix T is equivalent to an upper triangular matrix. We may state this fact in an equivalent way using invariant subspaces as follows:

If T is a complex-valued n by n matrix, there exist subspaces

$$\{0\} = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = \mathbf{C}^n$$

such that $T(M_k) \subseteq M_k$ for $k = 1, 2, \ldots, n$.

For another finite-dimensional example, the theory of Jordan canonical forms tells us that a linear transformation from $\mathbf{C}^n$ to itself can be expressed as a direct sum of matrices of the form

$$\begin{bmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & & \\ & & \lambda & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \lambda & 1 \\ & & & & & \lambda \end{bmatrix}.$$

The set of all invariant subspaces for any matrix block of this type forms a chain:

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_k.$$

For an example of the importance of invariant subspaces in the description of operators over infinite-dimensional Hilbert spaces, consider the *von Neumann algebra*, which is any weakly closed, unital, self-adjoint subalgebra of the algebra of operators on a Hilbert space. The von Neumann Double Commutant Theorem tells us that a von Neumann algebra is completely determined by its set of invariant subspaces [RR 73].

In this thesis, we will investigate more structures of invariant subspaces.

Definition 1.2 *If $\mathcal{S}$ is an arbitrary subset of $B(\mathcal{H})$, let $Lat(\mathcal{S})$ be the set of invariant subspaces for $\mathcal{S}$:*

$$Lat(\mathcal{S}) = \{M \subseteq \mathcal{H} : T(M) \subseteq M \ \forall T \in \mathcal{S}\}.$$

If $T \in B(\mathcal{H})$, then we define Lat(T) to be Lat($\{T\}$).

If $\mathcal{S}$ is a subspace of $B(\mathcal{H})$, then Lat($\mathcal{S}$) is a complete lattice. We call Lat($\mathcal{S}$) the lattice of invariant subspaces for $\mathcal{S}$. Note that Lat(T) is never empty because the trivial subspaces are in Lat(T) for every operator T. It is not known whether Lat(T) must always contain a nontrivial subspace. This is known to be true for some classes of operators , but the general question, called the *Invariant Subspace Problem*, remains unsolved today.

It is often convenient to think of the subspaces in terms of the projections onto their ranges. The following proposition is straightforward, but very useful.

Proposition 1.3 *If $T \in B(\mathcal{H})$ and P is a projection onto a subspace M, then $M \in Lat(T)$ if and only if $(1 - P)TP = 0$.*

It is also interesting to consider a parallel construction; given a set of subspaces of a Hilbert space, for which operators are these subspaces invariant?

Definition 1.4 *If $\mathcal{M}$ is a set of subspaces of $\mathcal{H}$, we define*

$$Alg(\mathcal{M}) = \{T \in B(\mathcal{H}) : T(M) \subseteq M \ \forall M \in \mathcal{M}\}.$$

It is not difficult to verify that Alg($\mathcal{M}$) is a unital subalgebra of $B(\mathcal{H})$ that is closed in the weak operator topology.

Definition 1.5 *We say a subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ is reflexive if $Alg(Lat(\mathcal{A})) = \mathcal{A}$.*

That is, a subalgebra $\mathcal{A}$ is reflexive if the set of operators which leave the invariant subspaces of $\mathcal{A}$ invariant is precisely $\mathcal{A}$ itself. We say an operator T is reflexive if the weakly closed algebra generated The term "reflexive" is attributed to Halmos; it appeared first in a paper by Radjavi and Rosenthal [RR 69].

There are many examples of reflexive algebras. D. Sarason proved that every commutative weakly closed algebra of normal operators or of analytic Toeplitz operators is reflexive [Sar 66]. J. Deddens showed that all isometries are reflexive[Ded 71]. Deddens and P. Fillmore [DF 75] have characterized the reflexive operators on finite-dimensional spaces as follows:

Theorem 1.6 *Let $\mathcal{H}$ be a finite-dimensional Hilbert space and let $T \in B(\mathcal{H})$. Then T is reflexive if and only if in the Jordan canonical form of the matrix for T, for each eigenvalue, the two biggest blocks have either the same dimension or their dimensions differ by one.*

Thus one simple example of a non-reflexive algebra is the algebra

$$\mathcal{A} = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in \mathbf{C} \right\}.$$

Elementary computations reveal that $\mathrm{Lat}(\mathcal{A}) = \left\{ \{0\}, span \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, \mathbf{C}^2 \right\}$ and

$$Alg(Lat(\mathcal{A})) = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathbf{C} \right\} \neq \mathcal{A}.$$

Note that this characterization can also be extended to algebraic operators (see [HN 82]). We can also define reflexivity for subspaces of operators.

Proposition 1.7 *Suppose $\mathcal{M}$ is a linear subspace of $B(\mathcal{H}, \mathcal{K})$. Then for an operator T in $B(\mathcal{H}, \mathcal{K})$, the following are equivalent. (See [Had 93].)*

1. *For all vectors x in $\mathcal{H}$, Tx belongs to the closure of $\mathcal{M}x$.*

2. *For every x in $\mathcal{H}$ and y in $\mathcal{K}$, if $y \perp \mathcal{M}x$, then $y \perp Tx$.*

3. *Whenever $A \in B(\mathcal{K})$, $B \in B(\mathcal{H})$ and $AVB = 0$ for every V in $\mathcal{M}$, then $ATP = 0$.*

4. *For every pair of projections (Q, P) in $\mathcal{P}_{\mathcal{K}} \times \mathcal{P}_{\mathcal{H}}$, if $Q\mathcal{M}P = 0$, then $QTP = 0$.*

The set of all such operators T in $B(\mathcal{H}, \mathcal{K})$ satisfying the above conditions is known as Ref$\mathcal{M}$. Whenever Ref$\mathcal{M} = \mathcal{M}$, we call $\mathcal{M}$ *reflexive.*

If $\mathcal{K} = \mathcal{H}$ and $\mathcal{M}$ is a unital subalgebra of $B(\mathcal{H})$, then Alg(Lat($\mathcal{M}$))=Ref$\mathcal{M}$. Many types of reflexivity have been defined. See D. Hadwin's paper [Had 93] for a unifying treatment of all of the various types of reflexivity.

Reflexive algebras such as von Neumann algebras, nest algebras, and CSL algebras have been studied in terms of sets of invariant projections. The aim of this thesis is to study reflexive subspaces in terms of sets of pairs of projections in an analogous fashion.

1.2 Coordinate Subspaces of Operator-Valued Matrices

Suppose M and N are closed subspaces of the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. Let S_1 be a subspace of the space of operators from M to N, S_2 a subspace of the space of operators from $M^\perp$ to N, S_3 a subspace of the space of operators from M to $N^\perp$, and S_4 a subspace of the space of operators from $M^\perp$ to $N^\perp$. Let $\mathcal{S}$ be the linear subspace of two by two operator-valued matrices in $B(\mathcal{H}, \mathcal{K})$ defined by

$$\mathcal{S} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A \in S_1, B \in S_2, C \in S_3, D \in S_4 \right\}.$$

In this section we will compare various properties of the subspaces S_i with the properties of $\mathcal{S}$. If each of the four S_i have a given property, must $\mathcal{S}$ also possess it? Conversely, which properties do the S_i inherit from $\mathcal{S}$?

1.2.1 Closure Properties

In the following subsections, the subspace $\mathcal{S}$ and the four coordinate subspaces are those defined in 1.2. Before proceeding, it is convenient to review some useful topologies on $B(\mathcal{H}, \mathcal{K})$ commonly used in operator theory.

The *weak operator topology* (WOT) on $B(\mathcal{H}, \mathcal{K})$ is the topology generated by the subbase $\{V_{x,y,\epsilon} : x \in \mathcal{H}, y \in \mathcal{K}, \epsilon > 0\}$ where

$$V_{x,y,\epsilon} = \{T \in B(\mathcal{H}, \mathcal{K}) : \|\langle Tx, y\rangle\| < \epsilon\}.$$

The *strong operator topology* (SOT) on $B(\mathcal{H}, \mathcal{K})$ is the topology of pointwise convergence generated by the subbase $\{V_{x,\epsilon} : x \in \mathcal{H}, \epsilon > 0\}$ where

$$V_{x,\epsilon} = \{T \in B(\mathcal{H}) : \|Tx\| < \epsilon\}.$$

The **-strong operator topology* (*-SOT) on $B(\mathcal{H}, \mathcal{K})$ is the topology in which convergence of a net $\{T_\lambda\}$ to an operator T occurs exactly when the net $\{T_\lambda\}$ converges to T in the strong operator topology in $B(\mathcal{H}, \mathcal{K})$ and the net $\{T_\lambda^*\}$ converges to T^* in the strong operator topology in $B(\mathcal{K}, \mathcal{H})$.

Proposition 1.8 *The subspace $\mathcal{S}$ is closed in the norm topology if and only if each S_i is closed in the norm topology. The result also holds if we replace the norm topology by the weak operator topology (WOT), the strong operator topology (SOT), or the *-strong operator topology.*

Proof. Because the inclusion map from $B(M, N)$ to $B(\mathcal{H}, \mathcal{K})$ is continuous, if $\mathcal{S}$ is closed, so is S_1 (for the *-SOT, take the adjoint and the appropriate inclusion map instead). Similarly, each of the other three subspaces are closed. Conversely, if each S_i is closed, consider $\mathcal{S}$ as a set of four-tuples rather than matrices. The topology on $\mathcal{S}$ is equivalent to the product topology on $S_1 \oplus S_2 \oplus S_3 \oplus S_4$, so $\mathcal{S}$ is closed if and only if each component is closed.

Proposition 1.9 *The span of the rank one operators in $\mathcal{S}$ is dense in the weak operator topology if and only if the span of the rank-one operators in S_i is dense in the weak operator topology for each i, $i = 1, 2, 3, 4$.*

Proof. If each S_i has this property and the matrix $\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ belongs to $\mathcal{S}$, then each A_i is a limit of rank-one operators. Thus

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix}.$$

is also in the closure of the span of the rank-one operators.

Suppose instead that $\mathcal{S}$ possesses this property. If A belongs to S_i, then $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ is a limit of a net of elements $\left\{ \begin{pmatrix} A_\lambda & B_\lambda \\ C_\lambda & D_\lambda \end{pmatrix} \right\}$ of S in the weak operator topology. Thus A itself is the WOT-limit of the net $\{A_\lambda\}$. Each of these A_n is a sum of rank-one operators, so the span of the rank-one operators is dense in S_1. Similarly, S_2, S_3, and S_4 have this property if $\mathcal{S}$ does.

Proposition 1.10 *Let n be a positive integer. Suppose $\mathcal{S}$ has the property that every rank n operator is the sum of n rank one operators in $\mathcal{S}$. Further suppose that F is a rank n*

operator in S_i for some $1 \leq i \leq 4$. Then F is a sum of n rank one operators in $\mathcal{S}_i$.

Proof. We will prove the result for $i = 2$ (which we will use later); the other three cases are as simple. Consider $\begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$, a rank n operator in $\mathcal{S}_2$. It must be the sum of n rank one operators in $\mathcal{S}_2$:

$$\begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}$$

Then

$$\begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.1}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(\sum_{i=1}^{4} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \right) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.2}$$

$$= \sum_{i=1}^{4} \begin{bmatrix} 0 & B_i \\ 0 & 0 \end{bmatrix} \tag{1.3}$$

Because each B_i has rank one or less, we are done.

An operator T in $B(\mathcal{H})$ is *compact* if the closure of the image of the unit ball of $\mathcal{H}$ is compact. We denote the set of all compact operators on $\mathcal{H}$ by $K(\mathcal{H})$. Since $K(\mathcal{H})$ is a norm-closed ideal in $B(\mathcal{H})$ that is also closed under adjoints, we may define the quotient C*-algebra $B(\mathcal{H})/K(\mathcal{H})$; this quotient is called the *Calkin algebra.*

Proposition 1.11 *The image of S in the Calkin algebra is closed if and only if the image of each S_i is closed in its corresponding Calkin algebra.*

Proof. In general, if the image of a set $\mathcal{A} \subseteq B(\mathcal{H})$ is closed in the Calkin algebra, then the set $\mathcal{A} + K(\mathcal{H})$ is closed in $B(\mathcal{H})$. Since $K(\mathcal{H}, \mathcal{K})$ is the set

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A \in K(M, N), B \in K(M^{\perp}, N), C \in K(M, N^{\perp}), D \in K(M^{\perp}, N^{\perp}) \right\}$$

$$= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A \in K_1, B \in K_2, C \in K_3, D \in K_4 \right\},$$

$\mathcal{S} + K(\mathcal{H}, \mathcal{K})$ is norm-closed in $B(\mathcal{H}, \mathcal{K})$ if and only if each $S_i + K_i$ is closed (Proposition 1.8).

1.2.2 Reflexivity

Suppose $\mathcal{M}$ is a linear subspace of $B(\mathcal{H}, \mathcal{K})$. Recall that Ref$\mathcal{M}$ is the set of all operators T in $B(\mathcal{H}, \mathcal{K})$ satisfying any (hence all) of the following:

1. For all vectors x in $\mathcal{H}$, Tx belongs to the closure of $\mathcal{M}x$.

2. For every x in $\mathcal{H}$ and y in $\mathcal{K}$, if $y \perp \mathcal{M}x$, then $y \perp Tx$.

3. Whenever $A \in B(\mathcal{K})$, $B \in B(\mathcal{H})$ and $A(\mathcal{M})B = 0$, $ATB = 0$.

4. For every pair of projections (Q, P) in $\mathcal{P}_{\mathcal{K}} \times \mathcal{P}_{\mathcal{H}}$, if $Q\mathcal{M}P = 0$, then $QTP = 0$.

Whenever Ref$\mathcal{M} = \mathcal{M}$, we call $\mathcal{M}$ *reflexive.*

Proposition 1.12 *Let $\mathcal{S}$ be defined as in Section 1.2. Then*

$$\mathrm{Ref}\mathcal{S} = \left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} : A_1 \in S_1, A_2 \in S_2, A_3 \in S_3, A_4 \in S_4 \right\}.$$

In particular, $\mathcal{S}$ is reflexive if and only if each of the S_i is reflexive.

Proof. Suppose

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in B(\mathcal{H}, \mathcal{K}). \tag{1.4}$$

Assume T belongs to Ref$\mathcal{S}$. We know that for every $\begin{pmatrix} x \\ y \end{pmatrix} \in M \oplus M^{\perp}$,

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \in \overline{\mathcal{S} \begin{pmatrix} x \\ y \end{pmatrix}}.$$

Substituting $y = 0$, we see that $T_1 x$ belongs to $\overline{S_1}$ and $T_3 x$ belongs to $\overline{S_3}$. Letting $x = 0$, we obtain parallel results for $T_2 y$ and $T_4 y$. Thus each T_i belongs to RefS_i for $i = 1, 2, 3, 4$.

Conversely, suppose T is written as above in equation 1.4 and assume that each T_i belongs to the corresponding RefS_i. To show that T belongs to Ref$\mathcal{S}$, take an arbitrary vector

$$\begin{pmatrix} x \\ y \end{pmatrix} \in M \oplus M^{\perp}.$$

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} T_1 x + T_2 y \\ T_3 x + T_4 y \end{pmatrix} \in \begin{pmatrix} \overline{S_1 x} + \overline{S_2 y} \\ \overline{ST_3 x} + \overline{S_4 y} \end{pmatrix} \subseteq \overline{\mathcal{S} \begin{pmatrix} x \\ y \end{pmatrix}}.$$

1.2.3 Hyperreflexivity

Definition 1.13 *A reflexive algebra $\mathcal{A}$ contained in $B(\mathcal{H})$ is hyperreflexive if there exists a constant k such that for all T in $B(\mathcal{H})$,*

$$dist(T, \mathcal{A}) \leq k \sup\{\|(1 - P)TP\| : P \in Lat(\mathcal{A})\}.$$

This term was introduced first by Arveson [Arv 84]. For ease of notation, we write

$$d(T,\mathcal{A}) = \sup\{||(1-P)TP|| : P \in \mathrm{Lat}(\mathcal{A})\}.$$

For a fixed algebra $\mathcal{A}$, the map d is a seminorm on $B(\mathcal{H})$. Thus $\mathcal{A}$ is a hyperreflexive algebra exactly when the two seminorms d and $dist$ are equivalent.

For a hyperreflexive algebra $\mathcal{A}$, we define the constant of hyperreflexivity, denoted by $k(\mathcal{A})$, as the smallest value of k for which definition 1.13 is true. If the algebra is not hyperreflexive, we say $k(\mathcal{A}) = \infty$. Later in this thesis, we will cite an example of a reflexive algebra that is not hyperreflexive. The next theorem is stated in greater generality in [Had 93].

Definition 1.14 *A reflexive subspace $\mathcal{S}$ contained in $B(\mathcal{H},\mathcal{K})$ is hyperreflexive if there exists a constant k such that for all T in $B(\mathcal{H},\mathcal{K})$,*

$$dist(T,\mathcal{S}) \leq k \sup\{||QTP|| : Q \in \mathcal{P}_{calK}, P \in \mathcal{P}_{calH}, Q\mathcal{S}P = 0\}.$$

Since the seminorms on a finite-dimensional Hilbert space having the same null set are equivalent, any reflexive subspace of operators on a finite-dimensional Hilbert space is hyperreflexive.

Proposition 1.15 *The subspace $\mathcal{S}$ defined on page 5 is hyperreflexive if and only if each S_i is hyperreflexive. The constants of hyperreflexivity are related as follows:*

$$k(S_i) \leq k(\mathcal{S}) \leq \sum_{i=1}^{4} k(S_i).$$

Proof. Assume $\mathcal{S}$ is hyperreflexive with constant of hyperreflexivity k. We will show here that S_3 is hyperreflexive; the proofs for the other three are analogous. Let W belong to $B(M, N^{\perp})$.

$$\begin{aligned}
dist(W, S_3) &= dist\left[\begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix}, \mathcal{S}\right] && (1.5)\\
&\leq kd\left[\begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix}, \mathcal{S}\right] && (1.6)\\
&= k \sup_{\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|=1} dist\left[\begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \overline{\mathcal{S}\begin{pmatrix} x \\ y \end{pmatrix}}\right] && (1.7)\\
&= k \sup_{\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|=1} \mathrm{dist}\left[\begin{pmatrix} 0 \\ Wx \end{pmatrix}, \overline{\begin{pmatrix} S_1x + S_2y \\ S_3x + S_4y \end{pmatrix}}\right] && (1.8)\\
&= k \sup_{\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|=1} \mathrm{dist}\left[Wx, \overline{S_3x + S_4y}\right] && (1.9)\\
&\leq k \sup_{\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|=1} \mathrm{dist}\left[Wx, \overline{S_3x}\right] && (1.10)\\
&\leq k \sup_{\|x\|=1} \mathrm{dist}\left[Wx, \overline{S_3x}\right] && (1.11)\\
&= k\, \mathrm{d}(W, S_3) && (1.12)
\end{aligned}$$

Thus S_3 is hyperreflexive with constant less than or equal to k.

For the converse, assume each S_i is hyperreflexive with constant k_i. Let T be an element of $B(\mathcal{H},\mathcal{K})$ written as in equation 1.4.

$$
\begin{aligned}
dist(T,\mathcal{S}) &= dist\left(\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}\right) && (1.13)\\
&\leq \sum_{i=1}^{4} dist(T_i, S_i) && (1.14)\\
&\leq \sum_{i=1}^{4} k_i d(T_i, S_i). && (1.15)
\end{aligned}
$$

Thus we need only show that $d(T_i, S_i) \leq d(T,\mathcal{S})$ for each i. We show this below for $i = 2$:

$$
d(T_2, S_2) = \sup_{||x||=1} dist(T_2 x, S_2 x)
$$

$$
= \sup_{\left\|\begin{pmatrix} x \\ 0 \end{pmatrix}\right\|=1} dist\left(\begin{bmatrix} 0 & T_2 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{bmatrix} 0 & S_2 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix}\right)
$$

$$
\leq \sup_{\left\|\begin{pmatrix} x \\ 0 \end{pmatrix}\right\|=1} dist\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix}\right)
$$

$$
\leq d(T, S).
$$

1.2.4 Properties D and D(r)

In their 1982 paper [HN 82], Hadwin and Nordgren define *property D* and *property D(r)*.

Definition 1.16 *A subspace $\mathcal{A}$ of $B(H)$ has property D if for every linear functional ϕ on $\mathcal{A}$ there exist vectors f and g in $\mathcal{H}$ so that $\phi(A) = \langle Af, g\rangle$ for every A in $\mathcal{A}$.*

Definition 1.17 *A subspace A has property D(r) if for every linear functional ϕ on A and for every $s > r$, there exists vectors f and g such that $\phi(A) = \langle Af, g\rangle$ and $\|f\|\|g\| \leq s\|\phi\|$.*

Every von Neumann algebra has property D (based on an idea in [Goo 66]). R. Olin and J. Thompson [OT 80] proved that every subnormal operator has property D(r) for some r. For a somewhat simpler example, we cite the following theorem of Hadwin and Nordgren.

Theorem 1.18 *If $T \in B(\mathcal{H})$ and T satisfies a polynomial equation of degree not exceeding 2, then T has property $D(\sqrt{10})$.*

For the subspace $\mathcal{S}$ defined on page 1.2, if each S_i has property D, it does not follow that $\mathcal{S}$ has property D. As an example, let each S_i be the Hilbert space of complex numbers for $i = 1, 2, 3, 4$. Then $\mathcal{S}$ is the set of two by two matrices with complex entries. It does not have property D; in particular, the trace cannot be expressed in the desired form.

If $\mathcal{S}$ has property D, then each S_i must have property D. Suppose ϕ is a linear functional on S_3. Then the linear functional can be extended to all of $\mathcal{S}$:

$$\tilde{\phi}\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right) = \phi(A_3).$$

There exist $m \in M$, $m' \in M^\perp$, $n \in N$, and $n' \in N^\perp$ so that

$$\tilde{\phi}\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right) = \left\langle \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{pmatrix} m \\ m' \end{pmatrix}, \begin{pmatrix} n \\ n' \end{pmatrix}\right\rangle = \phi(A_3)$$

Let A_1,A_2, and A_4 be zero operators. Then for every operator A_3 in S_3,

$$\phi(A_3) = \left\langle \begin{pmatrix} 0 \\ A_3 m \end{pmatrix}, \begin{pmatrix} n \\ n' \end{pmatrix} \right\rangle = \langle A_3 m, n' \rangle.$$

Note that by the Cauchy-Schwarz inequality, $\|\phi\| \leq \|f\|\,\|g\|$.

Suppose $r > 1$.

As in the case of property D, $\mathcal{S}$ may not have property D(r) even if every S_i has property D(r). If $\mathcal{S}$ has property D(r), each S_i must also possess this property. To see this, suppose ϕ is a linear functional on S_2, and $s > r$. We can extend ϕ to a linear functional $\tilde{\phi}$ on S as before and obtain vectors such that

$$\phi(A_2) = \tilde{\phi}\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right) = \left\langle \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{pmatrix} m \\ m' \end{pmatrix}, \begin{pmatrix} n \\ n' \end{pmatrix} \right\rangle$$

and $\left\|\begin{pmatrix} m \\ m' \end{pmatrix}\right\| \left\|\begin{pmatrix} n \\ n' \end{pmatrix}\right\| \leq s\|\tilde{\phi}\| = s\|\phi\|$. If we let A_1, A_3, and A_4 be zero operators, we obtain $\phi(A_2) = \langle A_2 m', n \rangle$ and $\|m'\|\|n\| \leq s\|\phi\|$.

1.3 The Main Construction

Consider the sets $\mathcal{P}_{\mathcal{H}}$ and $\mathcal{P}_{\mathcal{K}}$ of the projections onto closed subspaces of $\mathcal{H}$ and $\mathcal{K}$ respectively. From the usual partial orders (using set containment) on $\mathcal{P}_{\mathcal{K}}$ and $\mathcal{P}_{\mathcal{H}}$, we can define a partial order on $\mathcal{P}_{\mathcal{K}} \times \mathcal{P}_{\mathcal{H}}$ by $(Q_1, P_1) \leq (Q_2, P_2)$ if and only if $P_1 \leq P_2$ and $Q_2 \leq Q_1$. With this order, $(\mathcal{P}_{\mathcal{K}} \times \mathcal{P}_{\mathcal{H}}, \leq)$ is a complete lattice.

The map $\alpha : \mathcal{P}_{\mathcal{K}} \times \mathcal{P}_{\mathcal{H}} \longrightarrow \mathcal{P}_{\mathcal{K}\oplus\mathcal{H}}$ given by $\alpha(Q, P) = \begin{bmatrix} 1 - Q & \\ & P \end{bmatrix}$ is an order-preserving map with range

$$\left\{ \begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix} : P_1 \in \mathcal{P}_{\mathcal{K}}, P_2 \in \mathcal{P}_{\mathcal{H}} \right\}.$$

In particular, if $\mathcal{L} \subseteq \mathcal{P}_{\mathcal{K}\times\mathcal{H}}$ is a lattice, then the restriction of α to $\mathcal{L}$ onto its range is a lattice isomorphism. The map α is also a homeomorphism (using the product-SOT and the SOT respectively).

In the previous section, we defined $Alg(\mathcal{L})$ as the set of all operators taking each element of $\mathcal{L}$ into itself. If an operator $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in $B(\mathcal{H},\mathcal{K})$ belongs to $Alg(\alpha(\mathcal{L}))$, then for every pair (Q,P) in the lattice $\mathcal{L}$,

$$\begin{bmatrix} Q & 0 \\ 0 & 1-P \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1-Q & 0 \\ 0 & P \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Completing the computation, we see that for every pair (Q,P) in $\mathcal{L}$,

1. $QA(1-Q) = 0$;

2. $QBP = 0$;

3. $(1-P)C(1-Q) = 0$;

4. $(1-P)DP = 0$.

The upper left coordinate must belong to $Alg(\{Q : (Q,P) \in \mathcal{L}\})$ and the lower right coordinate must belong to $Alg(\{P : (Q,P) \in \mathcal{L}\})$. The upper right coordinate belongs to the subspace denoted below:

$$\mathcal{S}(\mathcal{L}) = \{T \in B(\mathcal{H},\mathcal{K}) : QTP = 0 \ \forall \ (Q,P) \in \mathcal{L}\}.$$

If we let $\hat{\mathcal{L}}$ represent the set $\{(1-P, 1-Q) : (Q,P) \in \mathcal{L}\}$, then the set of lower left coordinates is precisely $\mathcal{S}(\hat{\mathcal{L}})$.

In this paper, we will investigate the subspace $\mathcal{S}(\mathcal{L})$ for various types of lattices $\mathcal{L}$. Using the above map α, we will consider $Alg(\alpha(\mathcal{L}))$ as a subspace of two by two operator-valued matrices with $\mathcal{S}(\mathcal{L})$ imbedded in the upper right corner.

We previously considered what properties transfer between a subspace of operator-valued matrices and the component subspaces; we may use these results to compare $Alg(\alpha(\mathcal{L}))$ and $\mathcal{S}(\mathcal{L})$.

In the following sections, we examine particular types of lattices $\mathcal{L}$, namely nests and commutative subspace lattices. Many properties of the associated algebras can be used to describe the subspace $\mathcal{S}(\mathcal{L})$.

Chapter 2

Nest Algebras and Nest Subspaces

2.1 Nest Algebras

The nest algebra was developed to generalize the notion of the existence of triangular forms for finite-dimensional vector spaces. Suppose A is any linear transformation on an n-dimensional vector space. Then there exist invariant subspaces H_i for $i = 0, 1, 2, \ldots, n$ such that $\{0\} = H_0 \subset H_1 \subset H_2 \subset H_3 \subset \cdots \subset H_n = \mathcal{H}$ and for each i, the dimension of H_i is i. If we select a basis for $\mathcal{H}$ by starting with a basis for H_1 and extending it to a basis for H_2, then for H_3, and so on, we can write A as an upper triangular matrix (a_{ij}):

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & a_{nn} \end{bmatrix}$$

In general, this is not possible for arbitrary operators on Hilbert spaces. Operators need not have any one-dimensional invariant subspaces. Instead, we look for a chain of invariant subspaces (of varying dimensions) of a given operator.

Definition 2.1 *A nest $\mathcal{N}$ is a chain of closed subspaces of a Hilbert space $\mathcal{H}$ that contains both $\{0\}$ and $\mathcal{H}$ and is closed under intersection and closed span.*

In finite-dimensional algebras, it is natural to consider the algebra of upper triangular matrices with respect to a fixed orthonormal basis; this is precisely the algebra that takes the subspaces of a "maximal" nest of the Hilbert space to themselves. In a similar manner, we can define the nest algebra.

Definition 2.2 *The nest algebra $\tau(\mathcal{N})$ associated with a nest $\mathcal{N}$ of subspaces of a Hilbert space $\mathcal{H}$ is the set of all operators in $B(\mathcal{H})$ that leave every member of $\mathcal{N}$ invariant; that is, $T(N) \subseteq N$ for every N in $\mathcal{N}$.*

Note that $\tau(\mathcal{N})$ is indeed an algebra of operators. Ringrose began the study of nest algebras in the late 1950s [Rin 62]. The notation $\tau(\mathcal{N})$ was chosen because of the association with triangular matrices. If we view each member N of $\mathcal{N}$ as the corresponding projection $P = P(N)$ onto that subspace, we may write

$$\tau(\mathcal{N}) = \{T \in B(\mathcal{H}) : (1 - P)TP = 0 \ \forall P \in \mathcal{N}\}.$$

Several examples are given next:

1. Let μ denote Lebesgue measure and $L^2([0,1])$ denote the set of all functions that are square integrable over the unit interval. For each t in the interval $[0,1]$, let $N_t = \left\{f \in L^2([0,1]) : f|_{[t,1]} = 0 \ a.e.(\mu)\right\}$. Then $\mathcal{N} = \{N_t : t \in [0,1]\}$ is a nest.
2. Suppose $\mathcal{H}$ is a four-dimensional complex Hilbert space with orthonormal basis $\{e_1, e_2, e_3, e_4\}$. For $0 \le i \le 4$, let each P_i be the projection onto the subspace listed below:

$P_0 : \{0\}$

$P_1 : span\{e_1\}$

$P_2 : span\{e_1, e_2, e_3\}$

$P_3 : \mathcal{H}$.

The nest algebra $\tau(\mathcal{N})$ for the nest $\mathcal{N}$ consists of all four by four matrices having the following form:

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix}.$$

Nest algebras have been extensively examined. The book *Nest Algebras* by Ken Davidson is an excellent general resource [Dav 88].

2.2 Nest Subspaces

The study of nest algebras limits its focus to operators mapping a given Hilbert space to itself. Can some of the results obtained for nest algebras be extended to describe more general sets of operators in $B(\mathcal{H}, \mathcal{K})$? Instead of examining the set of operators taking a nest of subspaces to itself, we wish to consider the set of operators taking a nest of subspaces to another nest . Let β represent the map from $\mathcal{P}_{\mathcal{H}}$ to itself given by $\beta(P) = (1 - P, P)$. Note that β is an order-preserving map with respect to the order specified in the previous chapter. If we restrict β to a nest $\mathcal{N}$, we obtain as the image a complete lattice containing the pairs $(1,0)$ and $(0,1)$. By composing the maps α and β, we find that $\alpha(\beta(\mathcal{N}))$ is also a nest. Consider the map α introduced in section 1.3. What qualities must a subset of $\mathcal{P}_{\mathcal{H}} \times \mathcal{P}_{\mathcal{K}}$ possess so that its image under the map α is a nest?

Definition 2.3 *A chain $\mathcal{C}$ is a totally ordered subset of* $(\mathcal{P}_{\mathcal{H}} \times \mathcal{P}_{\mathcal{K}}, \leq)$ *that includes* $(1,0)$ *and* $(0,1)$ *and is closed under the product strong-operator topology.*

Nests are relatively simple examples of chains. A nest involves only one Hilbert space; furthermore, each pair of projections is an orthogonal pair. In contrast, not only can the coordinates exist in different spaces, but an entry in one of the coordinates may appear in more than one pair. We can also extend the notion of nest algebra to chains.

Definition 2.4 *Given a chain $\mathcal{C}$, its nest subspace $\tau(\mathcal{C})$ is the set of all operators B in $B(\mathcal{H},\mathcal{K})$ such that $QBP = 0$ for every pair of projections (Q,P) in $\mathcal{C}$.*

Lemma 2.5 *The set $\tau(\mathcal{C})$ is a linear subspace of $B(\mathcal{H},\mathcal{K})$ that is closed in the weak operator topology.*

Proof. Clearly $\tau(\mathcal{C})$ is closed under addition and scalar multiplication. Suppose $\{T_n\}$ is a net in $\tau(\mathcal{C})$ that converges to some T in $B(\mathcal{H},\mathcal{K})$. Let (Q,P) be a fixed member of $\mathcal{C}$ and let x belong to the range of P. We wish to show T_x is orthogonal to the range of Q. Let y belong to $ran(Q)$.

$$\langle y, Tx\rangle = \lim_n \langle y, T_n x\rangle = \lim_n \langle Qy, T_n Px\rangle = lim_n \langle y, QT_n Px\rangle = \lim_n \langle y, 0\rangle = 0.$$

Thus $QTP = 0$ for all pairs (Q,P) in $\mathcal{C}$ and T belongs to $\tau(\mathcal{C})$.

As stated above, every nest algebra is a nest subspace. The following examples show that a nest subspace need not be a nest algebra.

(i) Let $\mathcal{H}$ and $\mathcal{K}$ be finite-dimensional Hilbert spaces of dimensions h and k with sequences of closed subspaces $M = \{P_t : t = 0,1,2,\ldots,n\}$ and $N = \{Q_t : t = 0,1,2,\ldots,n\}$ respectively such that $\{0\} = P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n$ and $\{0\} = Q_0^\perp \subseteq Q_1^\perp \subseteq Q_2^\perp \subseteq$

$\cdots \subseteq Q_n^\perp$. Take an orthonormal basis for P_1 and extend it to P_2, then to P_3, and so on to obtain an orthonormal basis for $\mathcal{H}$. Similarly, obtain an orthonormal basis for $\mathcal{K}$ starting with $Q_1^\perp$. The nest algebra for the chain $\{(Q_t, P_t) : t = 0, 1, 2, \ldots, n\}$ written with respect to these orthonormal bases is the set of matrices with a given pattern of zeroes in the lower left corner. All finite-dimensional nest algebras can be written as block upper triangular matrices whose blocks on the diagonal are square; all finite-dimensional nest subspaces can be written as block upper triangular matrices whose blocks are rectangular (thus not necessarily symmetric with respect to the main diagonal).

As a particular example of a nest subspace, we consider the following: Let $\mathcal{H}$ be the complex Hilbert space of dimension six with orthonormal basis $\{e_n : n = 1,2,3,4,5,6\}$. Let $\mathcal{K}$ represent the complex Hilbert space of dimension seven with orthonormal basis $\{f_n : n = 1,2,3,4,5,6,7\}$. Suppose the ranges of the projections (Q_n, P_n) corresponding to the chain C are as specified below:

$P_0 : \{0\}$	$Q_0 : \mathcal{K}$
$P_1 : span\{e_1\}$	$Q_1 : span\{f_2, f_3, f_4, f_5, f_6, f_7\}$
$P_2 : span\{e_1, e_2\}$	$Q_2 : span\{f_2, f_3, f_4, f_5, f_6, f_7\}$
$P_3 : span\{e_1, e_2, e_3\}$	$Q_3 : span\{f_4, f_5, f_6, f_7\}$
$P_4 : span\{e_1, e_2, e_3, e_4, e_5\}$	$Q_4 : span\{f_5, f_6, f_7\}$
$P_5 : span\{e_1, e_2, e_3, e_4, e_5, e_6\}$	$Q_4 : span\{f_7\}$
$P_6 : \mathcal{H}$	$Q_6 : \{0\}$

The associated nest subspace is the set of all six by seven matrices T such that $Q_i T P_i = 0$ for $i = 0, 1, 2, \ldots, 6$. Thus the image of the vector $e_1 \in P_1$ under T must belong to the subspace orthogonal to Q_1, namely the span of the vector f_1. Continuing in this manner, we see that $\tau(C)$ is the set of all six by seven matrices with zeros in the entries specified below:

$$\begin{bmatrix} * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Compare the block structure of this nest subspace with that of the nest algebra in the previous section.

As previously suggested, one key idea for generalizing results about nest algebras is to associate a nest to a given chain. Let α be the map given in section 1.3 from $\mathcal{P}_{\mathcal{K}} \times \mathcal{P}_{\mathcal{H}}$ such that $\alpha((Q,P)) = (1-Q) \oplus P$. If $\mathcal{C}$ is a chain, then its image under α is a nest. Suppose T belongs to the nest $\mathcal{N} = \alpha(\mathcal{C})$. We may write T as a two by two matrix:

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in B(\mathcal{K})$, $B \in B(\mathcal{H},\mathcal{K})$, $C \in B(\mathcal{K},\mathcal{H})$ and $D \in B(\mathcal{H})$. For each projection in the nest, we have

$$\begin{bmatrix} Q_t & 0 \\ 0 & 1-P_t \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1-Q_t & 0 \\ 0 & P_t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} Q_tA(1-Q_t) & Q_tBP_t \\ (1-P_t)C(1-Q_t) & (1-P_t)DP_t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus T belongs to $\tau(\mathcal{N})$ if and only if each of the following hold:

1. A belongs to the nest algebra associated with the nest $\{1-Q_t : t\}$,

2. D belongs to the nest algebra associated with the nest $\{P_t : t\}$,

3. $Q_t B P_t = 0$ for all t, and

4. $(1 - P_t)C(1 - Q_t) = 0$ for all t.

For convenience, we will designate the nests $\{P_t : t\}$ and $\{1 - Q_t : t\}$ by $\mathcal{P}$ and $\mathcal{Q}^\perp$ respectively; the chain $\{(1 - P_t, 1 - Q_t) : t\}$ will be referred to as $\hat{\mathcal{C}}$.

2.3 Finite Rank Operators in Nest Subspaces

The finite rank operators of any given set or space are of interest because of their simplicity. We first need some preliminary definitions. For a pair (Q, P) in a chain $\mathcal{C}$, let $(Q, P)_-$ be the closed span of the union of pairs less than (Q, P); if the pair has an immediate predecessor, it is $(Q, P)_-$. Suppose no other pair in the chain contains Q as a first coordinate or P as a second coordinate. Then $(Q, P)_-$ consists of the projections P_- and $[(1 - Q)_-]$ in the nests on $\mathcal{H}$ and $\mathcal{K}$ respectively.

Every rank one operator in $B(\mathcal{H}, \mathcal{K})$ can be written in the form $x \otimes y$ for some $x \in \mathcal{K}$ and $y \in \mathcal{H}$ such that for every $z \in \mathcal{H}$, $(x \otimes y)(z) = \langle z, y \rangle x$. The rank one operators of a nest algebra have been characterized in a nice way.

Proposition 2.6 *Suppose $\mathcal{N}$ is a nest and $x \otimes y$ is a rank one operator in $\tau(\mathcal{N})$. Then there exists a subspace N in the nest such that x belongs to N and y belongs to $(N_-)^\perp$.*

Using our general construction, the finite-rank operators of a nest subspace are also easy to classify.

Proposition 2.7 *Let $x \otimes y$ be a rank one operator in $\tau(\mathcal{C})$. Then there exists a pair (Q, P) in $\mathcal{C}$ such that x belongs to $Q^\perp$ and y belongs to $(P_-)^\perp$.*

Proof. Assume $x \otimes y$ is a rank one operator in $\tau(\mathcal{C})$. Then the operator

$$\begin{bmatrix} 0 & x \oplus y \\ 0 & 0 \end{bmatrix}$$

is a member of the associated nest algebra $\tau(\mathcal{N}) = \alpha(\tau(\mathcal{C}))$. From the characterization of rank one operators in a nest algebra, there exists a projection $(1-Q) \oplus P$ in $\mathcal{N}$ such that $x \oplus 0$ belongs to $(1-Q) \oplus P$ and $0 \oplus y$ belongs to $[(1-Q) \oplus P]_-^\perp$. Thus x is in $Q^\perp$ and y is in $(P_-)^\perp$ as required.

In a 1968 paper, J. Erdos proved the following density theorem [Erd 68] :

Theorem 2.8 *The finite rank contractions of a nest algebra $\tau(\mathcal{N})$ are dense in the unit ball of $\tau(\mathcal{N})$ in the *-strong operator topology.*

To show this, he used the following proposition proved by Ringrose:

Proposition 2.9 *Let F be a rank n operator in $\tau(\mathcal{N})$. Then F is the sum of n rank one operators in $\tau(\mathcal{N})$.*

Using our general construction, we obtain the analogous results for nest subspaces:

Proposition 2.10 *Let n be a positive integer. Suppose F is a rank n operator in $\tau(\mathcal{C})$. Then F is a sum of n rank one operators in $\tau(\mathcal{C})$.*

Proof. Apply Propositions 1.10 and 2.9.

Proposition 2.11 *The finite-rank contractions in a nest subspace $\tau(\mathcal{C})$ are dense in the unit ball of $\tau(\mathcal{C})$ in the *-SOT.*

Proof. Let T be a contraction in $\tau(\mathcal{C})$ and consider the associated contraction $\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}$ in $\tau(\mathcal{N})$. From Erdos' result, there exist finite rank contractions in $\tau(\mathcal{N})$:

$$\begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \xrightarrow{*-\mathrm{SOT}} \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}$$

Hence in particular we have that for every h in $\mathcal{H}$,

$$\left\| \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \begin{bmatrix} 0 \\ h \end{bmatrix} - \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ h \end{bmatrix} \right\|^2 \longrightarrow 0.$$

Evaluating this expression,

$$\left\| \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \begin{bmatrix} 0 \\ h \end{bmatrix} - \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ h \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} B_t h - Th \\ D_t h \end{bmatrix} \right\|^2 = \|B_t h - Th\|^2 + \|D_t h\|^2,$$

we see that $\|B_t h - Th\|$ converges to zero for every h in $\mathcal{H}$.

2.4 Reflexivity and Hyperreflexivity

In a 1965 paper, Ringrose proved that all nest algebras are reflexive. Given a chain $\mathcal{C} = \{(Q_t, P_t) : t\}$, consider the associated nest $\alpha(\mathcal{C})$. Its nest algebra is reflexive and contains the nest subspace $\tau(\mathcal{C})$ as the set of upper right components. Applying Proposition 1.12, we see that all nest subspaces are also reflexive.

Arveson has shown that all nests are hyperreflexive with constant $k = 1$ in his well known Distance Formula. (See [Arv 74] or [Dav 88].)

Theorem 2.12 *Let $\mathcal{N}$ be a nest. For every operator A in $B(\mathcal{H})$,*

$$dist(A, \tau(\mathcal{N})) = \sup_{P \in \mathcal{N}} \|(1-P)AP\|.$$

Using Proposition 1.15 and Theorem 2.12, we deduce that nest subspaces are hyperreflexive with constant of hyperreflexivity $k = 1$.

Theorem 2.13 *Let C be a chain with associated nest $\mathcal{N}$. Suppose B belongs to $B(\mathcal{H}, \mathcal{K})$. Then*

$$\operatorname{dist}(B, \tau(C)) = \sup\{\|QBP\| : (Q, P) \in C\}$$

2.5 Relating Chains and Nests

Definition 2.14 *Let C_1 and C_2 be chains with associated nests $\mathcal{N}_1$ and $\mathcal{N}_2$. We say that the two chains are isomorphic if there exist unitaries U_1 in $B(\mathcal{K})$ and U_2 in $B(\mathcal{H})$ such that $C_2 = \{(U_1^* Q U_1, U_2^* P U_2) : (Q, P) \in C_1\}$.*

Proposition 2.15 *Let C_1 and C_2 be chains with associated nests $\mathcal{N}_1$ and $\mathcal{N}_2$.*

1. *The chain C_1 is isomorphic to the chain C_2 if the nest subspace $\tau(C_1)$ is isomorphic to $\tau(C_2)$.*

2. *If the nest subspace $\tau(C_1)$ is isomorphic to $\tau(C_2)$, then there exists a chain C such that $\tau(C) = \tau(C_2)$ and C is isomorphic to C_1.*

3. *If the chain C_1 is isomorphic to C_2, then $\tau(\mathcal{N}_1)$ is isomorphic to $\tau(\mathcal{N}_2)$.*

Proof. (1) Suppose the two chains are isomorphic and U_1 and U_2 are as above. Define $\phi : \tau(C_1) \longrightarrow \tau(C_2)$ by $\phi(T) = U_1^* T U_2$. It is easy to show that this map is linear, one-to-one, and onto.

(2) Now suppose $\tau(\mathcal{C}_1)$ and $\tau(\mathcal{C}_2)$ are unitarily equivalent. Then unitaries U_1 and U_2 exist such that $\tau(\mathcal{C}_2) = \{(U_1 T U_2) : T \in \mathcal{C}_1\}$. Let $\mathcal{C} = \{(U_1 Q U_1^*, U_2 P U_2^*) : (Q, P) \in \mathcal{C}_1\}$. By definition, $\mathcal{C}$ and $\mathcal{C}_1$ are isomorphic and

$$\begin{aligned} T \in \tau(\mathcal{C}_2) &\iff U_1^* T U_2 \in \tau(\mathcal{C}_1) \iff Q U_1^* T U_2 P = 0 \quad \forall (Q, P) \in \mathcal{C}_1 \\ &\iff U_1 Q U_1^* T U_2 P U_2^* = 0 \quad \forall (Q, P) \in \mathcal{C}_1 \\ &\iff T \in \tau(\mathcal{C}). \end{aligned}$$

Thus $\tau(\mathcal{C}) = \tau(\mathcal{C}_2)$ and $\mathcal{C}$ is isomorphic to $\mathcal{C}_1$.

(3) Suppose $\mathcal{C}_1$ amd $\mathcal{C}_2$ are isomorphic chains. Then there exist unitaries U_1 in $B(\mathcal{K})$ and U_2 in $B(\mathcal{H})$ such that $\mathcal{C}_2 = \{(U_1^* Q U_1, U_2^* P U_2) : (Q, P) \in \mathcal{C}_1\}$. Then

$$\mathcal{N}_2 = \left\{ \begin{bmatrix} U_1^*(1-Q)U_1 & \\ & U_2^* P U_2 \end{bmatrix} : (Q, P) \in \mathcal{C}_1 \right\} = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}^* \mathcal{N}_1 \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$$

Note that this is not the usual notion of equivalence of nests; the standard definition of isomorphic nests requires $U^* \mathcal{N}_1 = \mathcal{N}_2$ for some unitary U. Also, two equivalent nests need not have been derived from equivalent chains; in fact, a nest subspace can be derived from more that one chain. This suggests the following question: if $\mathcal{N}$ is an arbitarary nest, how can it be obtained from some chain $\mathcal{C}$?

Proposition 2.16 *Suppose $\mathcal{N}$ is an arbitrary nest on a Hilbert space $\mathcal{H}$. We can associate a chain C to $\mathcal{N}$ whenever there exists a projection R in $B(\mathcal{H})$ that commutes with every projection in $\mathcal{N}$.*

Proof. Suppose R is a projection in $B(\mathcal{H})$ that commutes with every projection in $\mathcal{N}$.

With respect to some orthonormal basis, we may write R as $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Elements of $\mathcal{N}$ must then be of the form $M_t = \begin{bmatrix} A_t & 0 \\ 0 & B_t \end{bmatrix}$ in which A_t and B_t are projections with ranges contained in ran$R$ and ran$(1 - R)$ respectively. Define a chain $\mathcal{C}$ by

$$\mathcal{C} = \{(1_R - A_t, B_t) : M_t \in \mathcal{N}\} .$$

Write $t \in \tau(\mathcal{N})$ as a two by two matrix with respect to ran$R$ and ran$(1 - R)$ and expand $(1 - M_t)TM_t$ to see that T decomposes in the desired manner.

2.6 The Radical

The Jacobson radical of a Banach algebra may be defined in several different ways. One common definition is given below:

Definition 2.17

$$\mathrm{rad}(\mathcal{A}) = \{T \in \mathcal{A} : TB \textit{ is quasinilpotent for every } B \in \mathcal{A}\} .$$

Theorem 2.18 *Given an element T in $\tau(\mathcal{N})$, the following are equivalent:*

1. *T belongs to rad$(\tau(\mathcal{N}))$.*

2. *The operator $1 + WT$ is left invertible for every W in $\tau(\mathcal{N})$.*

3. *(Ringrose) [Rin 65] For every $\epsilon > 0$ there exists a finite subnest F of $\mathcal{N}$ such that $||\triangle_F(T)|| < \epsilon$ where*

$$\triangle_F(T) = \sum_{P \in \mathcal{N}} (P - P_-)T(P - P_-).$$

We may use the last description to extend the concept of the radical to nest subspaces. For a chain $\mathcal{C}$ and an element B of $\tau(\mathcal{C})$, we say B belongs to $\mathrm{rad}(\tau(\mathcal{C}))$ if and only if for every $\epsilon > 0$ there is a finite subchain $\mathcal{C}_0$ of $\mathcal{C}$ such that the sum

$$\Delta_{\mathcal{C}_0}(B) = \sum_{(Q,P)\in\mathcal{C}_0} (Q - Q_+)B(P - P_-)$$

has norm less than ϵ.

Lemma 2.19 *An operatorB belongs to rad($\tau(\mathcal{C})$) if and only if* $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$ *belongs to rad($\tau(\mathcal{N})$).*

Proof. Omitted.

Corollary 2.20 *If B belongs to rad($\tau(\mathcal{C})$), then for every Y in $\tau(\hat{\mathcal{C}})$, $1 + YB$ has a left inverse in $\tau(\mathcal{P})$ and a right inverse in $\tau(\mathcal{Q}^\perp)$.*

Proof. By the above Lemma and Theorem, we know that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & WB \\ 0 & 1+YB \end{bmatrix}$$

has a left inverse $\begin{bmatrix} F & G \\ L & M \end{bmatrix}$. Since

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} F & G \\ L & M \end{bmatrix}\begin{bmatrix} 1 & WB \\ 0 & 1+YB \end{bmatrix} = \begin{bmatrix} F & FWB + G(1+YB) \\ L & LWB + M(1+YB) \end{bmatrix},$$

we see that $L = 0$. Hence $M(1 + YB) = 1$. Similarly, $1 + YB$ is right invertible in $\tau(\mathcal{Q}^\perp)$.

2.7 An Alternate Characterization of Nest Subspaces

In his paper *Another Description of Nest Algebras*, Deddens described nest algebras in terms of invertible operators. Every nest algebra consists of operators for which conjugation by powers of a fixed operator is norm-bounded. Suppose $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are invertible operators. Define

$$\mathcal{S}_{A,B} = \left\{X \in B(\mathcal{H},\mathcal{K}) : \exists M < \infty \text{ such that } \|B^k X A^{-k}\| < M \text{ for } k = 1,2,3,\ldots\right\}.$$

Proposition 2.21 *Suppose $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are invertible operators.*

1. *If $A_1 = TAT^{-1}$ and $B_1 = WBW^{-1}$ for $T \in B(\mathcal{H})$ and $W \in B(\mathcal{K})$, then $\mathcal{S}_{A_1,B_1} = W\mathcal{S}_{A,B}T^{-1}$.*
2. *$\mathcal{S}_{A^*,B^*} = (\mathcal{S}_{A,B})^*$.*
3. *$\mathcal{S}_{A,B} = \mathcal{S}_{|A|,|B|}$.*
4. *$(\mathcal{S}_{A,B})^* = \mathcal{S}_{(B^*)^{-1},(A^*)^{-1}}$.*

Proof.

$$\begin{aligned}
\mathcal{S}_{A_1,B_1} &= \left\{X \in B(\mathcal{H},\mathcal{K}) : \exists M < \infty \text{ such that } \left\|WB^kW^{-1}XTA^{-k}T^{-1}\right\| < M \text{ for } k = 0,1,2,\ldots\right\} \\
&= \left\{WYT^{-1} \in B(\mathcal{H},\mathcal{K}) : \exists M < \infty \text{ such that } \left\|WB^kYA^{-k}T^{-1}\right\| < M \text{ for } k = 0,1,2,\ldots\right\} \\
&= \left\{WYT^{-1} \in B(\mathcal{H},\mathcal{K}) : \exists M' < \infty \text{ such that } \left\|B^kYA^{-k}\right\| < M' \text{ for } k = 0,1,2,\ldots\right\} \\
&= W\mathcal{S}_{A,B}T^{-1}
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}_{A^*,B^*} &= \left\{X \in B(\mathcal{H},\mathcal{K}) : \exists M < \infty \text{ such that } \left\|(B^*)^k X (A^*)^{-k}\right\| < M \text{ for } k = 0,1,2,\ldots\right\} \\
&= \left\{X \in B(\mathcal{H},\mathcal{K}) : \exists M < \infty \text{ such that } \left\|A^{-k} X^* B^k\right\| < M \text{ for } k = 0,1,2,\ldots\right\} \\
&= \left\{Y^* \in B(\mathcal{H},\mathcal{K}) : \exists M < \infty \text{ such that } \left\|(B^*)^k Y (A^*)^{-k}\right\| < M \text{ for } k = 0,1,2,\ldots\right\} \\
&= (\mathcal{S}_{A,B})^*.
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}_{A,B} &= \left\{X \in B(\mathcal{H},\mathcal{K}) : \exists M < \infty \text{ such that } \left\|(B)^k X (A^{-k}\right\| < M \text{ for } k = 0,1,2,\ldots\right\} \\
&= \left\{X \in B(\mathcal{H},\mathcal{K}) : \exists M < \infty \text{ such that } \left\|W^k |B|^k X^* |A|^{-k} U^k\right\| < M \text{ for } k = 0,1,2,\ldots\right\} \\
&= \left\{Y^* \in B(\mathcal{H},\mathcal{K}) : \exists M < \infty \text{ such that } \left\||B|^k X^* |A|^{-k}\right\| < M \text{ for } k = 0,1,2,\ldots\right\} \\
&= \mathcal{S}_{|A|,|B|}.
\end{aligned}$$

Theorem 2.22 *For any two invertible operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$, the subspace*

$$\mathcal{S}_{A,B} = \left\{X \in B(\mathcal{H},\mathcal{K}) : \exists M < \infty \text{ such that } \|B^k X A^{-k}\| < M \text{ for } k = 1,2,3,\ldots\right\}.$$

is a nest subspace.

Proof. Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ be invertible operators. If T belongs to the set $\mathcal{S}_{A,B}$, then $\alpha(T)$ belongs to $\mathcal{B}_{B\oplus A}$. From Dedden's result, $\mathcal{B}_{B\oplus A}$ is the nest algebra generated by taking the spectral measure E for $B \oplus A$ and completing $\{E([0,a]) : a \geq 0\}$ to obtain the nest $\mathcal{N}(A)$. By the functional calculus, $E([0,a]) = \chi_{[0,a]}(B \oplus A)$ must also be a projection of the form

$$\begin{bmatrix} E_1[0,a] & 0 \\ 0 & E_2[0,a] \end{bmatrix}.$$

Thus we have a pair of spectral measures E_1 and E_2 giving rise to nests. Then $S_{A,B} = \tau(\{(1-E_1[0,a], 1-E_2[0,a]) : a \geq 0\})$ as shown below:

$$T \in S_{A,B} \longleftrightarrow \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \in \mathcal{B}_{B\oplus A} \tag{2.1}$$

$$\longleftrightarrow \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \in \tau(\mathcal{N}) \tag{2.2}$$

$$\longleftrightarrow (1-E([0,a]))\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} E([0,a]) = 0 \text{ for all } a \geq 0 \tag{2.3}$$

$$\longleftrightarrow (1-E_1([0,a]))\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} E_2([0,a]) = 0 \text{ for all } a \geq 0 \tag{2.4}$$

$$\longleftrightarrow T \in \tau(\{(1-E_1[0,a], 1-E_2[0,a]) : a \geq 0\}). \tag{2.5}$$

Is the converse true? Conjecture: *For every nest subspace $\tau(\mathcal{C})$ contained in $B(\mathcal{H},\mathcal{K})$, there exist invertible operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ such that $\tau(\mathcal{C}) = \mathcal{S}_{A,B}$.*

2.8 Multipliers of Nest Subspaces

For a nest algebra, the set of multipliers of the algebra is simply the nest algebra itself. For nest subspaces, we may speak of the set of left multipliers and the set of right multipliers. The set of left multipliers of a nest subspace $\tau(\mathcal{C})$ contains the nest algebra associated with the nest $\{1-Q : (Q,P) \in C\}$ because if A belongs to this nest algebra and $T \in \tau(C)$,

$$Q(AT)P = QATP - QA(QTP) = QA(1-Q)TP = [QA(1-Q)]TP = 0.$$

By examination, for any left multiplier A,

$$QA\rfloor_{\overline{\cup_{T\in\tau(C)} \text{ran } TP}} = 0.$$

This union cited in the above equation is contained in the range of $1-Q$, but the two need not be equal. A simple example is any nest subspace having two pairs (Q_1, P) and (Q_2, P) with $Q_1 \neq Q_2$. When this containment is equality for each pair (Q,P) in the nest, the set of left multipliers is the nest subspace generated by $\{1-Q : (Q,P) \in C\}$. Otherwise, the set of left multipliers is a nest subspace for the chain $\{(Q, \overline{\cup_{T\in\tau(C)} \text{ran } TP}) : (Q,P) \in C\}$.

For instance, let $\{e_1, e_2, e_3\}$ be an orthonormal basis for $\mathcal{H}$ and $\{f_1, f_2, f_3, f_4\}$ be a basis for $\mathcal{K}$. Let P_1 be the projection onto the span of $\{e_1\}$, Q_1 be the projection onto the span of $\{f_2, f_3\}$, and Q_2 the span of $\{f_3\}$. Let C be the chain given by $\{(1,0), (Q_1, P_1), (Q_2, P_1), (0,1)\}$. Then its nest subspace is the set of all 2 by 3 matrices with zero entries in the (2,1) and (3,1) positions. A direct calculation shows that the set of left multipliers is the set of all three by three matrices with zeroes in the same two positions. The nest algebra associated with the set $\{0, Q_1, Q_2, 1\}$ is the set of all upper triangular three by three matrices, a proper subset of the set of left multipliers.

Proposition 2.23 *Suppose that* C *is a chain that is purely atomic; that is, the two nests obtained from the coordinates of* C *are atomic. Suppose further that the chain contains no repeated projections. Then the set of left multipliers is the nest algebra* $\tau(\mathcal{Q}^{\perp})$.

Proof. Suppose that A is a left multiplier of a nest subspace of a chain C. Fix an arbitrary element (Q,P) of C. Select a nonzero element y from the orthogonal difference of the ranges of P and its predecessor. For every x belonging to the range of $1-Q$, $x\otimes y$ belongs to the nest subspace. We know $QA(x\otimes y)P=0$, so that $(QAx)\otimes Py=0$. Because Py is nonzero, QAx must be zero for every x in the range of $1-Q$. Hence $QA(1-Q)=0$ as desired.

Note that analagous results are easy to obtain to describe the set of right multipliers of a nest subspace.

Proposition 2.24 *Suppose $A\in B(\mathcal{K})$ and $B\in B(\mathcal{H})$ are operators such that $ATB=0$ for every T in the nest subspace $\tau(C)$. Then for some pair (Q,P) in the chain, $QA=A$ and $PB=B$.*

Proof. Form the associated nest $\alpha(C)$. From a known result for nest algebras, there is a projection R in the associated nest such that $(1-R)(A\oplus 0)=(A\oplus 0)$ and $R(0\oplus B)=(0\oplus B)$. Thus we obtain a pair (Q,P) in the chain yielding the desired result.

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & 1-P \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} QA & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 1-Q & 0 \\ 0 & P \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & PB \end{bmatrix}$$

Theorem 2.25 *Suppose ϕ is a linear transformation on $\tau(C)$ for a chain $C=\{(Q_i,P_i):i\in I\}$ with the property that for every $S\in\tau(C)$, there exists an operator T_S in $\tau(\{Q_i^\perp : i\in I\})$ such that $\phi(S)=T_S S$. Then there exists $T\in\tau(C)$ such that for every $S\in\tau(C)$, $\phi(S)=TS$.*

Proof. Suppose $y\otimes x\in\tau(C)$. Then there exists $i\in I$ such that $y\in Q_i^\perp$ and $x\in(P_i^\perp)_-$.

By the hypotheses, there exists an operator $T_{y,x} \in \tau(\{Q_i^\perp : I \in I\})$ with $\phi(y \otimes x) = T_{y,x}(y \otimes x) = (T_{y,x}y) \otimes x$. For each fixed index i, define the map $\alpha_x : Q_i^\perp \longrightarrow Q_i^\perp$ by $\alpha_x(y) = T_{y,x}y$. Then $\phi(y \otimes x) = (T_{y,x}y) \otimes x = \alpha_x(y) \otimes x$. The map α_x is linear:

$$\phi((y_1 + \lambda y_2) \otimes x) = [\alpha_x(y_1 + \lambda y_2)] \otimes x \tag{2.6}$$

$$\text{and} \tag{2.7}$$

$$\phi((y_1 + \lambda y_2) \otimes x) = \phi((y_1 \otimes x) + (\lambda y_2 \otimes x)) \tag{2.8}$$

$$= \phi(y_1 \otimes x) + \lambda\phi(y_2 \otimes x) \tag{2.9}$$

$$= (\alpha_x y_1 \otimes x) + \lambda(\alpha_x y_2 \otimes x) \tag{2.10}$$

$$= (\alpha_x y_1 + \lambda\alpha_x y_2 \otimes x) \tag{2.11}$$

Thus $[\alpha_x(y_1 + y_2)] \otimes x = (\alpha_x y_1 + \alpha_x y_2 \otimes x)$ and $\alpha_x(y_1 + y_2) = \alpha_x y_1 + \alpha_x y_2$.

The map ϕ_x is also bounded:

$$\begin{aligned} ||\phi(y \otimes x)|| &\leq ||\phi|| ||y \otimes x|| = ||\phi|| ||y|| ||x|| \\ ||\phi(y \otimes x)|| &= ||\alpha_x(y) \otimes x|| = ||\alpha_x(y)|| ||x|| \end{aligned}$$

Thus $||\alpha_x(y)|| \leq ||\phi|| \, ||y||$ and α_x is bounded with norm less than $||\phi||$.

Since ϕ is a linear map,

$$\phi(y \otimes (x_1 + x_2) = \phi(y \otimes x_1) + \phi(y \otimes x_2) \tag{2.12}$$

$$\alpha_{x_1+x_2}(y) \otimes (x_1 + x_2) = \alpha_{x_1}(y) \otimes x_1 + \alpha_{x_2}(y) \otimes +x_2 \tag{2.13}$$

$$\alpha_{x_1+x_2}(y) \otimes x_1 + \alpha_{x_1+x_2}(y) \otimes x_2 = \alpha_{x_1}(y) \otimes x_1 + \alpha_{x_2}(y) \otimes +x_2 \tag{2.14}$$

$$[\alpha_{x_1+x_2}(y) \otimes x_1] - [\alpha_{x_1}(y) \otimes x_1] = \alpha_{x_2}(y) \otimes +x_2 - \alpha_{x_1+x_2}(y) \otimes x_2 \tag{2.15}$$

$$(\alpha_{x_1+x_2}(y) - \alpha_{x_1}(y)) \otimes x_1 = (\alpha_{x_2}(y) - \alpha_{x_1+x_2}(y)) \otimes x_2 \tag{2.16}$$

Suppose x_1 and x_2 are orthogonal and nonzero.

$$[(\alpha_{x_1+x_2}(y) - \alpha_{x_1}(y)) \otimes x_1](x_1) = [(\alpha_{x_2}(y) - \alpha_{x_1+x_2}(y)) \otimes x_2](x_1) \tag{2.17}$$

$$(\alpha_{x_1+x_2}(y) - \alpha_{x_1}(y)) \|x_1\|^2 = 0 \tag{2.18}$$

$$\alpha_{x_1+x_2}(y) = \alpha_{x_1}(y). \tag{2.19}$$

Similarly, $\alpha_{x_1+x_2}(y) = \alpha_{x_2}(y)$. Because $\alpha_{x_1}(y) = \alpha_{x_2}(y)$ for all x_1 and x_2 in an orthogonal basis for $(P_i^\perp)_-$, $\alpha_x = \alpha_{x'}$ for every x and x' in $(P_i^\perp)_-$. Then we may define a single map α on each $(P_i^\perp)_-$ by $\phi(y \otimes x) = \alpha(y) \otimes x$. Since the $(P_i^\perp)_-$ are nested, $\phi(y \otimes x) = \alpha(y) \otimes x$ on all rank one tensors $x \otimes y$.

2.9 Cleaning

Although we would like a chain to be closed in the product strong operator topology, we wish to eliminate pairs in the chain that add no information in determining the nest subspace. Such pairs are an unnecessary complication in many proofs. The "cleaning" procedure is outlined below:

1. Consider each set of pairs (Q_i, P) with the same second projection P. From each of these sets, omit all of the pairs from the chain except the one with the largest Q_i (the least of these pairs).

2. Complete the chain by taking its closure in the product strong operator topology.

3. Now consider all sets of pairs of the form (Q, P_j). Delete all of the pairs with the same Q except the one with the largest P_j (the largest of these pairs).

4. Complete the chain again.

For a chain with finitely many pairs, no repeats will remain; for a chain over an infinite-dimensional space, no more than two pairs will have the a given left coordinate or given right coordinate.

Chapter 3

Commutative Subspace Lattices, Algebras and Subspaces

3.1 Commutative Subspace Lattices

A commutative subspace lattice (CSL) is a complete lattice of subspaces $\mathcal{L}$ such that the set of projections corresponding to each subspace is abelian. In particular, the reflexive algebras having a commutative subspace lattice, called CSL lattices, are of interest. Every nest algebra is a CSL algebra. For another example, consider the following set $\mathcal{L} = \{P_0, P_1, P_2, P_3\}$ of commuting projections in $M_2(C)$:

$$P_0 = 0$$

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $Alg(\mathcal{L}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : a, d \in C \right\}$ is a CSL algebra. What does a CSL algebra with a finite lattice look like in general?

Suppose $\{P_i : i = 1, 2, \ldots, n\}$ is a commutative subspace lattice. Examine all of the following products of the form

$$\prod_{\substack{k_i \in \{0,1\} \\ i = 1, 2, \ldots, n}} P_i^{k_i}$$

where $P_i^0 = P_i^{\perp}$ and $P_i^1 = P_i$. Since the lattice is commutative, each of these 2^n products is a projection. They are mutually orthogonal and their sum is the identity, so we may use bases for each of the new (nonzero) projections to obtain an orthonormal basis. With respect to this basis, each P_i can be written as a diagonal matrix whose entries consist only of zeroes and identity operators. The CSL algebra can then be envisioned as an algebra of square matrices of dimension $\leq 2^n$ in which certain entries must be zero.

Commutative subspace lattices on separable Hilbert spaces were described in 1974 by Arveson [Arv 74]. In order to understand this description, we need a few definitions.

Suppose X is a compact metric space with a finite regular Borel measure m. Let $\{f_n\}$ be a countable collection of continuous real-valued functions on X. Define a pre-order $\leq$ by

$$x \leq y \longleftrightarrow f_n(x) \leq f_n(y) \text{ for all positive integers } n.$$

This type of ordering is called a *standard pre-order* on X.

A Borel subset E of X is called an *increasing set* if whenever x belongs to E and $x \leq y$, then y belongs to E.

Let

$$\mathcal{L}(X, \leq, m) = \{f \in L^2(X, m) : supp(f) \text{ is an increasing set}\}.$$

This can be verified to be a complete lattice closed in the SOT. In a sense, these are the only commutative subspace lattices on separable Hilbert spaces.

Theorem 3.1 *Let $\mathcal{L}$ be a commutative subspace lattice on a separable Hilbert space. Then $\mathcal{L}$ is unitarily equivalent to some $\mathcal{L}(X, \leq, m)$ where X is a compact metric space, m is a regular Borel measure on X, and $\leq$ is a standard pre-order.*

3.2 CSL Subspaces and Pattern Subspaces

Suppose the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ can be decomposed as direct sums $\mathcal{H} = \bigoplus_{i=1}^{n} \mathcal{H}_i$ and $\mathcal{K} = \bigoplus_{j=1}^{m} \mathcal{K}_j$. Then any operator $T \in B(\mathcal{H}, \mathcal{K})$ can be thought of as an m by n operator-valued matrix (T_{ij}) such that $T_{ij} \in B(H_i, K_j)$ for each pair (i, j).

Definition 3.2 *We say ρ is an (m by n) pattern in case ρ is a subset of the cartesian product $\{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$.*

As convenient notation, we will represent patterns as m by n matrices with asterisks at the entries with coordinate pairs belonging to ρ and zeros at the entries whose coordinates do not belong to ρ. (Note that we also may include in the definition the possibility of infinite patterns.)

Definition 3.3 *If ρ is an m by n pattern, then the complementary pattern ρ' is the pattern* $[\{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}] - \rho$.

Thus if a matrix represents a pattern, to represent its complementary pattern, we simply replace all the asterisks by zeros and and zeros by asterisks.

Definition 3.4 *Suppose $\mathcal{H} = \bigoplus_{i=1}^{n} \mathcal{H}_i$ and $\mathcal{K} = \bigoplus_{j=1}^{m} \mathcal{K}_j$ are Hilbert spaces. A* pattern subspace *$\mathcal{S}(\rho)$ of an m by n pattern ρ is the set of all operators $T = (T_{ij}) \in B(\mathcal{H}, \mathcal{K})$ such that for all (i, j) not in ρ, $T_{ij}=0$.*

Hence a member of the pattern subspace $\mathcal{S}(\rho)$ looks like the matrix representing ρ with the asterisks replaced by operators in the appropriate spaces.

Definition 3.5 *We will call $\mathcal{L} = \{(Q_i, P_i) : i\}$ contained in $\mathcal{P}_{\mathcal{K}} \times \mathcal{P}_{\mathcal{H}}$ a* family of commuting pairs *if $\mathcal{L}$ is closed in the product strong-operator topology and for every pair of indices i and j, $Q_iQ_j = Q_jQ_i$ and $P_iP_j = P_jP_i$.*

Note that if $\mathcal{L}$ is a family of commuting pairs $\{(Q_i, P_i) : i\}$, then $\alpha(\mathcal{L}) = \{(1-Q_i) \oplus P_i : i\}$ is a commutative subspace lattice. This motivates the next definition.

Definition 3.6 *If $\mathcal{L}$ is a family of commuting pairs as specified in definition 3.5, we define the CSL subspace $\mathcal{S}(\mathcal{L})$ to be the subspace of $B(\mathcal{H}, \mathcal{K})$ given by*

$$\{T \in B(\mathcal{H}, \mathcal{K}) : QTP = 0 \text{ for every pair } (Q, P) \in \mathcal{L}\}.$$

If we write elements of the CSL algebra generated by $\alpha(\mathcal{L})$ as two by two matrices, then $\mathcal{S}(\mathcal{L})$ consists of the entries which lie in the upper right corner.

By performing a construction similar to that of page 40 separately on the sets $\{P_i : (Q_i, P_i) \in \mathcal{L}\}$ and $\{Q_i^\perp : (Q_i, P_i) \in \mathcal{L}\}$, we may think of any CSL subspace having a finite lattice as a pattern subspace. Conversely, every pattern subspace can be realized as a CSL subspace. For an m by n pattern ρ, whenever a pair (i,j) does not belong to ρ, include the pair (Q,P) in which Q is the m by m matrix projection with a one on the ith entry of the diagonal and P is the n by n matrix with a one on the jth entry of the diagonal. Form a lattice using these pairs, which commute as needed (since they are all diagonal).

Theorem 3.7 *Suppose $\mathcal{L}$ is a family of commuting pairs $\{(Q_i, P_i) : i\}$ contained in $\mathcal{K} \times \mathcal{H}$ where $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces. Let $\mathcal{V}$ be the CSL subspace associated with $\mathcal{L}$. Then there exists a sequence of pattern subspaces $\{\mathcal{V}_i\}$ such that*

1. $\mathcal{V}_1 \supseteq \mathcal{V}_2 \supseteq \mathcal{V}_3 \supseteq \ldots$ *and*

2. $\mathcal{V} = \bigcap_{n=1}^{\infty} \mathcal{V}_n$.

Proof. Since $\mathcal{L}$ is commutative and closed in the strong operator topology, we may choose a SOT-dense sequence (Q_n, P_n) in $\mathcal{L}$ such that $\mathcal{V} = \mathcal{S}(\{(Q_n, P_n) : n \text{ is a positive integer}\}$. For each positive integer n, let $\mathcal{V}_n = \mathcal{S}\{(Q_1, P_1), (Q_2, P_2), \ldots, (Q_n, P_n)\}$. Then $\mathcal{V} = \bigcap \mathcal{V}_n$.

Unlike the class of nest algebras, the CSL algebras need not be hyperreflexive. K. R. Davidson and S. C. Power [DP 84] first constructed an example of a non-hyperreflexive CSL algebra. This construction involves the concept of dual product, which was created by D. R. Larson (See [Lar 86]). We first cite some results and definitions necessary to understand the dual product.

As a review, we cite the definitions of tensor product for Hilbert spaces and for operators as defined in [KR 83]:

Theorem 3.8 (page 132) *Suppose that $\mathcal{H}_1, \ldots, \mathcal{H}_n$ are Hilbert spaces. (i) There is a Hilbert space $\mathcal{H}$ and a weak Hilbert-Schmidt mapping $p : \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathcal{H}$ with the following property: given any weak Hilbert-Schmidt mapping L from $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ into a Hilbert space $\mathcal{K}$, there is a unique bounded linear mapping T from $\mathcal{H}$ to $\mathcal{K}$, such that $L = Tp$; moreover, $\|T\| = \|L\|_2$. (ii) If $\mathcal{H}'$ and p' have the same properties attributed in (i) to $\mathcal{H}$ and p, there is a unitary transformation u from $\mathcal{H}$ onto $\mathcal{H}'$ such that $p' = Up$. (iii) If $v_m, w_m \in \mathcal{H}_m$ and Y_m is an orthonormal basis of $\mathcal{H}_m (m = 1, 2, \ldots, n)$, then*

$$\langle p(v_1, \ldots, v_n), p(w_1, \ldots, w_n)\rangle = \langle v_1, w_1\rangle \cdots \langle v_n, w_n\rangle,$$

the set $\{p(y_1, \ldots, y_n) : y_1 \in Y_1, \ldots, y_n \in Y_n\}$ is an orthonormal basis of $\mathcal{H}$, and $\|p\|_2 = 1$.

We denote $\mathcal{H}$, the (Hilbert) tensor product, as $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. The vector $p(v_1, \ldots, v_n)$ in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. is usually denoted by $x_1 \otimes \cdots \otimes x_n$. The span of such vectors form an everywhere dense subspace of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$.

Proposition 3.9 (page 144) *If $\mathcal{H}_1, \ldots, \mathcal{H}_n$ and $\mathcal{K}_1, \ldots, \mathcal{K}_n$ are Hilbert spaces and $A_m \in B(\mathcal{H}_m, \mathcal{K}_m)$ $(m = 1, 2, \ldots, n)$, there is a unique bounded linear operator A from $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ into $\mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_n$ such that*

$$A(x_1 \otimes \cdots \otimes x_n) = A_1 x_1 \otimes \cdots \otimes A_n x_n$$

whenever $x_1 \in \mathcal{H}_1, \ldots, x_n \in \mathcal{H}_n$.

Definition 3.10 *Suppose $\mathcal{S}$ is a subspace of the dual space X^* of a Banach space X. Then the annihilator of $\mathcal{S}$ is*

$$\mathcal{S}_\perp = \bigcap_{\phi \in \mathcal{S}} \ker \phi = \{x \in X : \forall s \in S, \langle x, s\rangle = 0\}.$$

The preannihilator of $\mathcal{S}$ is defined to be

$$\{\phi \in X^{**} : \forall s \in \mathcal{S}, \phi(s) = 0\}.$$

Since the dual of the trace class operators

$$\mathcal{C}_1(\mathcal{H}) = \text{span}\left\{T \in B(\mathcal{H}) : T \geq 0, \left\{\sum_i \langle Te_i, e_i\rangle : \{e_i\} \text{ is an orthonormal basis for } \mathcal{H}\right\} < \infty\right\}$$

is $B(\mathcal{H})$, given a matrix $A = (a_{ij})$ in $B(\mathcal{H})$ and a matrix $K = (b_{ij})$ in $\mathcal{C}_1(\mathcal{H})$, the form mentioned above can be defined as

$$\langle A, K\rangle = \text{trace}(AK^{transpose}) = \sum a_{ij} b_{ij}.$$

This is the quantity found by taking the Schur (elementwise) product of the matrices A and K and summing all of the entries.

Let $\mathcal{V}$ and $\mathcal{W}$ be weak-* closed subspaces of $B(\mathcal{H})$ and $B(\mathcal{K})$ respectively. Let $\mathcal{V}_\perp$ and $\mathcal{W}_\perp$ represent the pre-annihilators of $\mathcal{V}$ and $\mathcal{W}$ in the corresponding sets of trace-class operators. Then we define $\mathcal{V} * \mathcal{W}$, called the *dual product* of $\mathcal{V}$ and $\mathcal{W}$, by

$$\begin{aligned} \mathcal{V} * \mathcal{W} &= (\mathcal{V}_\perp \otimes \mathcal{W}_\perp)^\perp && (3.1)\\ &= \{A \in B(\mathcal{H} \otimes \mathcal{K}) : tr(A(X \otimes Y)) = 0 \ \forall X \in \mathcal{V}_\perp, \forall Y \in \mathcal{W}_\perp\}. && (3.2)\end{aligned}$$

To better understand the dual product, it is helpful to examine some relatively simple cases.

(1) Suppose $\mathcal{V}$ is a pattern subspace of the three by two matrices with pattern

$$\begin{bmatrix} 0 & * & 0 \\ * & * & 0 \end{bmatrix}$$

and $\mathcal{W}$ is a pattern subspace of the two by two matrices with pattern

$$\begin{bmatrix} * & 0 \\ * & * \end{bmatrix}.$$

Then the pre-annihilator of $\mathcal{V}$ is the set of all (trace-class) matrices whose Schur product (element-wise multiplication) with the elements of $\mathcal{S}$ is zero. For pattern subspaces, this is precisely the complementary pattern:

$$\begin{bmatrix} * & 0 & * \\ 0 & 0 & * \end{bmatrix}.$$

Similarly, the pre-annihilator of $\mathcal{W}$ can be shown to be

$$\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}.$$

To evaluate the spatial tensor product of the pre-annihilators, insert the pattern for $\mathcal{V}_\perp$ into each nonzero component of $\mathcal{W}_\perp$ and resize each zero in the pattern to the appropriate dimensions:

$$(\mathcal{V}_\perp \otimes \mathcal{W}_\perp) = \left[\begin{matrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} * & 0 & * \\ 0 & 0 & * \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{matrix} \right].$$

To finish taking the dual product, we now find the annihilator. Again we consider those matrices whose Schur product with the given quantity is zero:

$$\mathcal{V} * \mathcal{W} = (\mathcal{V}_\perp \otimes \mathcal{W}_\perp)^\perp = \left[\begin{matrix} \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix} & \begin{pmatrix} 0 & * & 0 \\ * & * & 0 \end{pmatrix} \\ \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix} & \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix} \end{matrix} \right].$$

(2) Let D_3 represent the set of diagonal three by three matrices over $\mathbf{C}$:

$$D_3 = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \in \mathbf{C} \right\}.$$

If $\mathcal{S}$ is any weak-* closed subspace of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, then by the Hahn-Banach Theorem, $(\mathcal{S}_\perp)^\perp = \mathcal{S}$. Following the same steps above, the dual product of $\mathcal{S}$ and D_3 is

$$\mathcal{S} * D_3 = \left\{ \begin{bmatrix} X_1 & S_{12} & S_{13} \\ S_{21} & X_2 & S_{23} \\ s_{31} & S_{32} & X_3 \end{bmatrix} : X_i \in B(\mathcal{H}), S_{ij} \in \mathcal{S} \right\}.$$

Davidson and Power use the latter example in the construction of a non-hyperreflexive CSL algebra. First, they prove that if $\mathcal{S}$ is a proper reflexive subspace of $B(\mathcal{H})$, then

$$k(\mathcal{S} * D_3) \geq (9/8)^{1/2} k(\mathcal{S}).$$

Thus if $\mathcal{S}_n$ represents the dual product of n copies of D_3, then

$$k(\mathcal{S}_n) \geq (9/8)^{n/2}.$$

Each of the $\mathcal{S}_n$ is a pattern subspace since the dual product of two pattern subspaces is a pattern subspace. We quote the proof of Davidson and Power both for historical reasons and to show how pattern CSL subspaces simplify the construction of a non-hyperreflexive CSL algebra. We have emphasized the part of the text we will use in our alternate proof.

Theorem 3.11 *There is a CSL algebra that is not hyperreflexive.*

Proof. (From *Nest Algebras* by Davidson, page 369.) *Let $\mathcal{S}_n$ be as above, and set $\mathcal{S} = \sum_{n\geq 1} \bigoplus \mathcal{S}_n$ to be the weak-* closure of the direct sum.* Let $\mathcal{D} = \sum_{n\geq 1} \bigoplus \mathcal{D}_{3^n}$ be the diagonal algebra, and note that $\mathcal{S}$ is a $\mathcal{D}$ bimodule. The algebra $\mathcal{A}$ consisting of all operators of the form $\begin{bmatrix} D_1 & S \\ 0 & D_2 \end{bmatrix}$ such that D_1 and D_2 belong to $\mathcal{D}$ and S belongs to $\mathcal{S}$ is a weak-* closed algebra containing the atomic masa $\mathcal{D} \bigoplus \mathcal{D}$. Hence $\mathcal{L} = Lat(\mathcal{A})$ is an atomic CSL. It is readily apparent that $\mathcal{A}$ contains the rank one operators of the form PKQ where P and Q are rank-one projections in $\mathcal{D} \bigoplus \mathcal{D}$ and K belongs to $\mathcal{A}$. These are easily seen to span a weak-* dense subspace of Alg($\mathcal{L}$). So $\mathcal{A}$ is reflexive, and hence is a CSL algebra.

By the easy parts of Exercises 24.2 and 24.3,

$$k(\mathcal{A}) \geq K(\mathcal{S}) \geq \sup_{n\geq 1} k(\mathcal{S}_n) = \infty.$$

So $\mathcal{A}$ is not hyperreflexive. [End of Proof.]

Notice that the operator $\mathcal{S}$ is still a CSL subspace since it is a direct sum of pattern subspaces. Therefore $\mathcal{S}$ is an example of a CSL subspace that is not hyperreflexive. Applying Proposition 1.15 to the CSL algebra associated with $\mathcal{S}$ gives us another example of a non-hyperreflexive CSL algebra.

3.3 Dual Products of CSL Subspaces

We have shown that every CSL subspace is the intersection of a decreasing sequence of pattern subpaces. We would like to know that the converse is also true. If this indeed is the case, we can then prove that the dual product of two CSL subspaces is a CSL subspace.

Proposition 3.12 *Suppose $\mathcal{R}$ and $\mathcal{T}$ are pattern subspaces. Then the dual product $\mathcal{R} * \mathcal{T}$ is also a pattern subspace.*

Definition 3.13 *The tensor product of two patterns P_1 (m_1 by n_1) and P_2 (m_2 by n_2) is the (m_1m_2) by (n_1n_2) pattern $\{([c-1]m_1 + a, [d-1]n_1 + b) : (a,b) \in P_1, (c,d) \in P_2\}$ whenever m_1 and n_1 are finite.*

The matrix for $P_1 \otimes P_2$ can be obtained by inserting a (m_1 by n_1) matrix of zeros into each zero position of P_2 and the pattern P_1 in to replace the nonspecified entries of P_2.

Proof of Proposition 3.12. Suppose $\mathcal{R}$ is derived from pattern P_1 and $\mathcal{T}$ is derived from pattern P_2;that is, $\mathcal{R} = \mathcal{S}(P_1)$ and $\mathcal{T} = \mathcal{S}(P_2)$. Then the pre-annihilators $\mathcal{R}_\perp$ and $\mathcal{T}_\perp$

are the pattern subspaces $\mathcal{S}(P_1')$ and $\mathcal{S}(P_2')$ respectively. The tensor product of these two pre-annihilators is the pattern subspace of the tensor product $\mathcal{S}(P_1' \otimes P_2')$. Finally, to get the dual product of $\mathcal{R}$ and $\mathcal{T}$, we take the annihilator of this tensor product to obtain that $\mathcal{R} * \mathcal{T} = \mathcal{S}((P_1' \otimes P_2')')$, a pattern subspace.

Theorem 3.14 *Suppose $\mathcal{S}$ and $\mathcal{T}$ are CSL subspaces such that $\mathcal{S} = \bigcap_n \mathcal{S}_n$ and $\mathcal{T} = \bigcap_n \mathcal{T}_n$ as the intersections of decreasing sequences of pattern subspaces. Then $\mathcal{S} * \mathcal{T} = \bigcap_n (\mathcal{S}_n * \mathcal{T}_n)$. Thus the dual product of any two CSL subspaces is a CSL subspace.*

Proof.

$$\begin{aligned}
(\mathcal{S}_\perp \otimes \mathcal{T}_\perp) &= \left(\bigcap_n \mathcal{S}_n\right)_\perp \otimes \left(\bigcap_n \mathcal{T}_n\right)_\perp && (3.3)\\
&= \overline{\left(\bigcup_n (\mathcal{S}_n)_\perp\right)} \otimes \overline{\left(\bigcup_n (\mathcal{T}_n)_\perp\right)} && (3.4)\\
&= \overline{\bigcup_n [(\mathcal{S}_n)_\perp \otimes (\mathcal{T}_n)_\perp]} && (3.5)
\end{aligned}$$

Taking the annihilator, we obtain the desired result:

$$\begin{aligned}
(\mathcal{S}_\perp \otimes \mathcal{T}_\perp)^\perp &= \overline{\bigcup_n [(\mathcal{S}_n)_\perp \otimes (\mathcal{T}_n)_\perp]}^\perp && (3.6)\\
&= \left[\bigcup_n (\mathcal{S}_n)_\perp \otimes (\mathcal{T}_n)_\perp\right]^\perp && (3.7)\\
&= \bigcap_n \left[(\mathcal{S}_n)_\perp^\perp \otimes (\mathcal{T}_n)_\perp^\perp\right] && (3.8)\\
&= \bigcap_n (\mathcal{S}_n \otimes \mathcal{T}_n) && (3.9)
\end{aligned}$$

Since $\{\mathcal{S}_n \otimes \mathcal{T}_n\}$ is itself a decreasing sequence of pattern subspaces, we have shown that $\mathcal{S} * \mathcal{T}$ is the intersection of a decreasing sequence of pattern subspaces. Thus the dual product of two CSL subspaces is a CSL subspace if the following conjecture is true:

Conjecture: The intersection of a decreasing sequence of pattern subspaces is a CSL subspace.

Given a sequence $\mathcal{S}_1 \supseteq \mathcal{S}_2 \supseteq \cdots$ of pattern subspaces, it would be desirable to associate to each pattern subspace $\mathcal{S}_n$ a family of commuting pairs $\mathcal{L}_n$ in a canonical way such that the pairs of all the families all commute (perhaps even so that $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}_3 \subseteq \cdots$). Then $\bigcup_n \mathcal{L}_n$ is a family of commuting pairs and

$$\bigcap_n \mathcal{S}_n = \mathcal{S}\left(\bigcup_n \mathcal{L}_n\right).$$

The proof hinges on finding the canonical lattice associated with each pattern subspace.

3.4 Complete Distributivity

Some results are known relating the lattice structure of certain commutative subspace lattices with the topological structure of the compact operators in the algebra generated by the lattices. It is desirable to find analogous results for CSL subspaces.

Definition 3.15 *A lattice $\mathcal{L}$ is* distributive *if*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

for all a, b, and c belonging to the lattice $\mathcal{L}$ and

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

These two distributive laws are equivalent, so checking only one of them will generally suffice. It is easily checked that all commutative subspace lattices are distributive.

Definition 3.16 *A complete lattice $\mathcal{L}$ is* completely distributive *if the distributive laws hold for arbitrary sets.*

Let $\{\mathcal{M}_\lambda : \lambda \in \Lambda\}$ be a collection of subsets of $\mathcal{L}$ and let $\prod \mathcal{M}_\lambda$ represent the collection of functions $f : \Lambda \longrightarrow \cup\mathcal{M}_\lambda$ with $f(\lambda) \in \mathcal{M}_\lambda$. The distributive laws become

$$\bigwedge_{\lambda\in\Lambda} \left(\bigvee \mathcal{M}_\lambda\right) = \bigvee_{f\in\prod \mathcal{M}_\lambda} \left(\bigwedge_{\lambda\in\Lambda} f(\lambda)\right)$$

and

$$\bigvee_{\lambda\in\Lambda} \left(\bigwedge \mathcal{M}_\lambda\right) = \bigwedge_{f\in\prod \mathcal{M}_\lambda} \left(\bigvee_{\lambda\in\Lambda} f(\lambda)\right)$$

for all $\{\mathcal{M}_\lambda \subseteq \mathcal{L} : \lambda \in \Lambda\}$.

As before, one of these laws is enough to describe a completely distributive lattice.

Recall that given any separable Hilbert space $\mathcal{H}$ and a fixed operator $T \in B(\mathcal{H}, \mathcal{K})$, it can be shown that the quantity

$$\sum_n \|Te_n\|^2$$

is a constant (finite or infinite) independent of the choice of orthonormal basis $\{e_n\}$. If this constant is finite, we say that the operator T is a *Hilbert-Schmidt operator.* The set of all such operators is an ideal in $B(\mathcal{H}, \mathcal{K})$.

Theorem 3.17 *Suppose $\mathcal{L}$ is a commutative subspace lattice. Then the following are equivalent:*

1. *The lattice $\mathcal{L}$ is completely distributive.*
2. *The span of the rank-one operators of Alg($\mathcal{L}$) is dense in Alg($\mathcal{L}$) in the weak* topology.* *[HLM 84]*

3. The Hilbert-Schmidt operators of Alg($\mathcal{L}$) are dense in Alg($\mathcal{L}$) in the weak operator topology. [LL 83]

Suppose $\mathcal{L}$ is a family of commuting pairs as earlier defined. What does the above theorem tell us about $\mathcal{L}$ and the structure of $\mathcal{S}(\mathcal{L})$?

Suppose a family of commuting pairs $\mathcal{L} = \{(Q_i, P_i) : i\}$ is completely distributive.

Since the map α is a lattice isomorphism, we know $\alpha(\mathcal{L}) = (1 - Q_i, P_i)$ is also completely distributive. Suppose $\mathcal{L}$ is a completely distributive family of commuting pairs and T belongs to the CSL subspace generated by $\mathcal{L}$. Then there exists a net $\left\{ \begin{bmatrix} A_\lambda & B_\lambda \\ C_\lambda & D_\lambda \end{bmatrix} \right\}$ of Hilbert-Schmidt operators converging to $\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}$ in the weak operator topology. This means that for every $x \in \mathcal{H}$ and $y \in \mathcal{K}$,

$$\left\langle \begin{bmatrix} A_\lambda & B_\lambda \\ C_\lambda & D_\lambda \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix}, \begin{bmatrix} y \\ 0 \end{bmatrix} \right\rangle \longrightarrow \left\langle \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix}, \begin{bmatrix} y \\ 0 \end{bmatrix} \right\rangle.$$

That is, $\langle B_\lambda x, y\rangle \longrightarrow \langle Tx, y\rangle$. Since the set of all Hilbert-Schmidt operators form an ideal in $B(\mathcal{H}, \mathcal{K})$, the operators in the net $\{B_\lambda\}$ are Hilbert-Schmidt operators. Thus we have:

Proposition 3.18 *If a family of commuting pairs is completely distributive, then the Hilbert-Schmidt operators of the corresponding CSL subspace are dense in that CSL subspace in the weak operator topology.*

Finally, the proof of the following is left to the reader:

Proposition 3.19 *If a family of commuting pairs is completely distributive, then the span of the rank-one operators of the corresponding CSL subspace is dense in that CSL subspace in the weak* topology.*

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